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The regulated four-parameter one-dimensional point interaction

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Abstract. The general four-parameter point interaction in one-dimensional quantum mechanics is regulated. It allows the exact solution, but not the perturbative one. We conjecture that this is due to the interaction not being asymptotically free. We then propose a different breakup of unperturbed theory and interaction, which now is asymptotically free but leads to the same physics. The corresponding regulated potential can be solved both exactly and perturbatively, in agreement with the conjecture.

1. Introduction

The easiest point interaction in one-dimensional quantum mechanics was introduced by Fermi more than 60 years ago [1]. It corresponds to a Dirac δ , and its mathematical interpretation came almost thirty years later [2]. It is now an almost standard example in elementary quantum mechanics. In one dimension, however, and only in one dimension, there exists a much more complex point interaction, which in its most general form depends on four real parameters [3]. One of these corresponds to the so-called δ' potential, an interaction surrounded with confusion, controversy and issues of interpretation [4–8]. There are of course no difficulties if the problem is solved in terms of boundary conditions which make sure that the Hamiltonian is self-adjoint. Even its Brownian measure has been constructed [9]. A deeper physical understanding, however, requires a regulated potential which, when the regulator is eventually removed, leads to the same physics. Surprisingly it does not seem to exist in the published literature, and it even has been put forward that the problem does not allow regulation. The nearest one has come to this is, on the one hand, a regulated Hamiltonian, which is not self-adjoint, although it converges to one which is self-adjoint when the short distance cutoff is removed [10]; and on the other, the proof that a regulator exists for three parameters, which include the so-called δ^2 [11]. This, to a physicist, is not a satisfactory state of the art.

Here we give a complete solution to the problem. We find a regulated potential which corresponds to a self-adjoint Hamiltonian and reproduces all the four-parameter physics. However, we then show that it does not allow us to solve the problem perturbatively, i.e. it leads to a perturbation theory which is not renormalizable. We argue that this is because the interaction is not asymptotically free, as there is scattering at infinite energy. We then reformulate the problem by proposing a different partition of the unperturbed

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Hamiltonian and the interaction. The new regulated potential now allows both exact and perturbative solutions. This is consistent with our conjecture that asymptotic freedom is necessary for perturbative renormalizability in quantum mechanics, as the new interaction is asymptotically free.

We start with a section where the physics of the four-parameter boundary conditions, as well as all the limits and particular cases, are reviewed. For the next two sections, which contain all the results, readers not used to the language of quantum field theory as applied to quantum mechanics may find [12] helpful. Our units are $\hbar = 2m = 1$.

2. The boundary conditions and its physics

The most general point interaction in one dimension is described by a free Hamiltonian on the real line with the origin excluded where the boundary values of the wavefunction and its derivative satisfy the constraints [9]:

$$\begin{pmatrix} -\psi'_L \\ \psi'_R \end{pmatrix} = \begin{pmatrix} \rho + \alpha & -\rho e^{i\theta} \\ -\rho e^{-i\theta} & \rho + \beta \end{pmatrix} \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} \quad (2.1)$$

which ensures self-adjointness of the Hamiltonian. Here $\rho \geq 0$, α , β and $0 \leq \theta < 2\pi$ are real parameters and the subindices indicate whether the value of the wavefunction or its derivative, which both have to be finite, correspond to the boundary of the negative halfline (L) or the positive halfline (R). Equation (2.1) can also be written, for $\rho > 0$, as

$$\begin{pmatrix} \psi'_R \\ \psi_R \end{pmatrix} = e^{-i\theta} \begin{pmatrix} 1 + \beta/\rho & \alpha + \beta + \alpha\beta/\rho \\ 1/\rho & 1 + \alpha/\rho \end{pmatrix} \begin{pmatrix} \psi'_L \\ \psi_L \end{pmatrix} \quad (2.2)$$

which is more adequate for taking the $\rho \rightarrow \infty$ limit.

Note that equations (2.1) and (2.2) are invariant under

$$\begin{aligned} \psi_R &\leftrightarrow \psi_L \\ \psi'_R &\leftrightarrow -\psi'_L \end{aligned} \quad (2.3)$$

which corresponds to the parity operation, $x \leftrightarrow -x$, together with

$$\begin{aligned} \alpha &\leftrightarrow \beta \\ \theta &\leftrightarrow -\theta \end{aligned} \quad (2.4)$$

which will therefore correspond to changing the sign of the antisymmetric interaction. This implies violation of parity invariance, as long as $\alpha \neq \beta$ or $\theta \neq 0$.

For $\rho = 0$, equation (2.1) reduces to

$$\begin{aligned} -\psi'_L &= \alpha\psi_L \\ \psi'_R &= \beta\psi_R \end{aligned} \quad (2.5)$$

which corresponds to a $L^2(\mathbb{R}^-)$ and $L^2(\mathbb{R}^+)$ problem respectively, unrelated to each other, with no probability flowing from one to the other. The parameter θ becomes irrelevant. If, furthermore, $\alpha = \beta = 0$ we have

$$\psi'_L = \psi'_R = 0 \quad (2.6)$$

which are Neumann boundary conditions on both halflines.

For $\alpha = \beta$, $\theta = 0$, equation (2.1) becomes

$$\begin{aligned}\psi'_R - \psi'_L &= \alpha(\psi_R + \psi_L) \\ \psi'_R + \psi'_L &= (2\rho + \alpha)(\psi_R - \psi_L)\end{aligned}\quad (2.7)$$

which corresponds to a symmetric interaction. The first condition refers to the even wavefunction, and the second to the odd. If furthermore $\alpha = 0$ then

$$\begin{aligned}\psi'_R &= \psi'_L \\ \psi'_L &= \rho(\psi_R - \psi_L)\end{aligned}\quad (2.8)$$

which corresponds to the ill-named (it is symmetric!) δ' interaction, which only acts on odd wavefunctions. If finally $\rho \rightarrow \infty$ there is no interaction,

$$\begin{aligned}\psi'_R &= \psi'_L \\ \psi_R &= \psi_L.\end{aligned}\quad (2.9)$$

For $\rho \rightarrow \infty$ and $\theta = 0$, (2.2) reduces to

$$\begin{aligned}\psi_R &= \psi_L \\ \psi'_R - \psi'_L &= (\alpha + \beta)\psi_L\end{aligned}\quad (2.10)$$

which is the δ interaction. The parameter $\alpha - \beta$ is irrelevant. It is also a symmetric interaction which only acts on even wavefunctions.

For $\rho \rightarrow \infty$ and $\alpha + \beta = 0$ equation (2.2) becomes

$$\begin{aligned}\psi'_R &= e^{-i\theta}\psi'_L \\ \psi_R &= e^{-i\theta}\psi_L\end{aligned}\quad (2.11)$$

which shows that θ is just a constant phase shift in crossing the origin. The breaking of time reversal invariance due to the noninvariance of the boundary conditions under complex conjugation is made clear in these last equations, but is seen already in equation (2.1).

For $\rho > 0$, if two bound states exist their energies are given by

$$\sqrt{-E_0} = -\rho - \frac{1}{2}(\alpha + \beta) \pm \frac{1}{2}\sqrt{4\rho^2 + (\alpha - \beta)^2} > 0. \quad (2.12)$$

They do not depend on θ . When only one bound state exists its energy is given by the upper sign expression of (2.12). For $\rho = 0$ at most one bound state exists, and its energy is

$$\sqrt{-E_0} = -\alpha > 0 \quad (2.13)$$

which requires $\alpha = \beta < 0$, so that the eigenvalues in \mathbb{R}^+ and \mathbb{R}^- are the same.

For the scattering states, $E \equiv k^2 > 0$, and defining the scattering amplitudes according to

$$\psi(x) = e^{ikx} + ie^{ik|x|}(f_+(k)\theta(x) + f_-(k)\theta(-x)) \quad (2.14)$$

with $\theta(x) = 0$ for $x < 0$ and $\theta(x) + \theta(-x) = 1$, we have

$$\begin{aligned} f_+(k) &= i + \frac{2k\rho}{\Delta} e^{-i\theta} \\ f_-(k) &= \frac{-i}{\Delta} (k^2 - ik(\alpha - \beta) + \rho(\alpha + \beta) + \alpha\beta) \end{aligned} \quad (2.15)$$

where

$$\Delta \equiv k^2 + ik(2\rho + \alpha + \beta) - \rho(\alpha + \beta) - \alpha\beta. \quad (2.16)$$

The high-energy limit of the scattering amplitudes is

$$\begin{aligned} \lim_{k \rightarrow \infty} f_+(k) &= i \\ \lim_{k \rightarrow \infty} f_-(k) &= -i \end{aligned} \quad (2.17)$$

so that there is scattering even at infinite energy, and the wavefunction becomes

$$\lim_{k \rightarrow \infty} \psi(x) = 2 \cos(kx) \theta(-x) \quad (2.18)$$

which corresponds to total reflection with Neumann boundary condition.

At low energies

$$\begin{aligned} \lim_{k \rightarrow 0} f_+(k) &= i \\ \lim_{k \rightarrow 0} f_-(k) &= i \end{aligned} \quad (2.19)$$

and

$$\lim_{k \rightarrow 0} \psi(x) = 2i \sin(kx) \theta(-x) \quad (2.20)$$

which also corresponds to total reflection, but with Dirichlet boundary condition.

For $\rho = 0$ equation (2.15) becomes

$$\begin{aligned} f_+(k) &= i \\ f_-(k) &= -i \frac{k - i\alpha}{k + i\alpha} \end{aligned} \quad (2.21)$$

which again correspond to total reflection. The parameters β and θ are then of course irrelevant.

For $\alpha = \beta = \theta = 0$ we obtain

$$f_+(k) = i \frac{k}{k + 2i\rho} = -f_-(k) \quad (2.22)$$

which is the δ' scattering amplitude.

For $\rho \rightarrow \infty$, $\theta = 0$ one obtains the δ scattering amplitude,

$$f_+(k) = -i \frac{\alpha + \beta}{2ik - (\alpha + \beta)} = f_-(k). \quad (2.23)$$

Note that now the scattering amplitudes vanish at high energies, i.e. the limits $\rho \rightarrow \infty$ and $k \rightarrow \infty$ do not commute.

The optical theorem reads

$$2 \operatorname{Im} f_+(k) = |f_+(k)|^2 + |f_-(k)|^2. \quad (2.24)$$

It will be convenient to introduce

$$\begin{aligned} f_s(k) &\equiv \frac{1}{2}(f_+(k) + f_-(k)) \\ f_a(k) &\equiv \frac{1}{2}(f_+(k) - f_-(k)). \end{aligned} \quad (2.25)$$

One can check that for a symmetric interaction, i.e. $\alpha = \beta$, $\theta = 0$, the optical theorem holds for the symmetric and antisymmetric scattering amplitudes

$$\begin{aligned} \operatorname{Im} f_s(\alpha = \beta, \theta = 0) &= |f_s(\alpha = \beta, \theta = 0)|^2 \\ \operatorname{Im} f_a(\alpha = \beta, \theta = 0) &= |f_a(\alpha = \beta, \theta = 0)|^2 \end{aligned} \quad (2.26)$$

which allows a straightforward introduction of phase shifts. This is because a symmetric interaction in one dimension is equivalent to a rotationally invariant interaction in more dimensions.

3. The strong regulated potential

From the work of Carreau [10] we are led to consider the following potential:

$$\begin{aligned} V(x) &= \theta(x + \epsilon)\theta(\epsilon - x) \left(W(\epsilon) + 2iR(\epsilon) \frac{d}{dx} \right) + (B(\epsilon) - iR(\epsilon))\delta(x - \epsilon) \\ &\quad + (A(\epsilon) + iR(\epsilon))\delta(x + \epsilon) \end{aligned} \quad (3.1)$$

where $W(\epsilon)$, $R(\epsilon)$, $A(\epsilon)$ and $B(\epsilon)$ are real functions of the short distance regulator $\epsilon > 0$, which eventually is taken to be zero. The terms which contain $R(\epsilon)$ are imaginary and violate time reversal invariance. They will take care of the θ parameter and depend on the momentum. The Hamiltonian

$$H = -\frac{d^2}{dx^2} + V(x) \quad (3.2)$$

is self-adjoint, and although $V(x)$ contains Dirac deltas it is sufficiently regulated to allow for its exact solution without further regulation. One can regulate the Dirac deltas too, but

this complicates the analysis unnecessarily. One can then show that one recovers (2.1) in the limit $\epsilon \rightarrow 0$ if

$$\begin{aligned} W(\epsilon) &= \frac{1}{4}\rho^2 f^2(\rho\epsilon) \\ R(\epsilon) &= -\frac{\theta}{2\epsilon} \\ A(\epsilon) &= -\frac{1}{2}\rho f(\rho\epsilon) + \rho + \alpha \\ B(\epsilon) &= -\frac{1}{2}\rho f(\rho\epsilon) + \rho + \beta \end{aligned} \tag{3.3}$$

where the function $f(x)$ is given by the small x condition

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{\exp(xf(x))}{f(x)} &= 1 \\ \lim_{x \rightarrow 0} xf(x) &\rightarrow \infty \end{aligned} \tag{3.4}$$

which, for small x , implies

$$f(x) \sim \frac{-\ln x + \ln(-\ln x)}{x}. \tag{3.5}$$

Note that terms subdominant with respect to the ones shown in (3.3) are irrelevant in the $\epsilon \rightarrow 0$ limit.

For finite ρ one does not need the function $f(x)$ beyond equation (3.5), but if one is interested in the limit $\rho \rightarrow \infty$ one would need an $f(x)$ such that

$$\begin{aligned} \lim_{x \rightarrow \infty} \frac{\exp(xf(x))}{f(x)} &= 1 \\ \lim_{x \rightarrow \infty} xf(x) &\rightarrow \infty \end{aligned} \tag{3.6}$$

but no solution to this system exists, so that the regulated potential $V(x)$ is only valid for finite ρ . The $\rho \rightarrow \infty$ limit has to be taken after the continuum limit $\epsilon \rightarrow 0$ is taken. Note that for the $\rho \rightarrow 0$ limit, $W(\epsilon)$, $A(\epsilon)$ and $B(\epsilon)$ also diverge, so that the $\rho \rightarrow 0$ limit also has to be taken after the $\epsilon \rightarrow 0$ limit. Finally, if one is interested in high energies, also the $k \rightarrow \infty$ limit has to be taken after the regulator is removed, $\epsilon \rightarrow 0$, as $V(x)$ leads to no scattering at high energies, while equation (2.1) shows scattering at high energies. The reason why $V(x)$ does not scatter at high enough energy is that it corresponds to wells and barriers of finite depth and height (recall that the Dirac δ 's can be regularized as well), which can be neglected as compared to the kinetic energy as k becomes larger and larger. The presence of a contribution linear in k does not spoil the argument. In other words, at high energy the physics is determined by the regulator, and we are only interested in regulator-independent physics.

The potential is symmetric for $\alpha = \beta$, $\theta = 0$ as then $A(\epsilon) = B(\epsilon)$ and $R(\epsilon) = 0$. It would be antisymmetric for $W(\epsilon) = 0$ and $A(\epsilon) = -B(\epsilon)$, but no values of the parameters ρ , α , β and θ allow for an antisymmetric potential. This is why a genuine δ' does not exist.

Consider now a perturbative expansion based on the Lippmann–Schwinger equation for the Hamiltonian of (3.2) with $V(x)$ given by (3.1), but with $W(\epsilon)$, $R(\epsilon)$, $A(\epsilon)$ and $B(\epsilon)$ to be fixed in such a way that as $\epsilon \rightarrow 0$ one reproduces the perturbative expansion of the scattering amplitudes $f_+(k)$ and $f_-(k)$ given in (2.15) and (2.16). As the free theory corresponds to $\alpha = \beta = \theta = 0$, $\rho \rightarrow \infty$, one expands for small α , β and θ and large ρ . To first order one obtains

$$\begin{aligned} f_+^{(1)}(k) &= -\theta - \frac{\alpha + \beta}{2k} + \frac{k}{2\rho} \\ f_-^{(1)}(k) &= -\frac{\alpha + \beta}{2k} - \frac{k}{2\rho}. \end{aligned} \quad (3.7)$$

One can see that the second order term is imaginary.

From the Lippmann–Schwinger equation

$$\psi(x) = e^{ikx} - \int dy G_k(x-y)V(y)\psi(y) \quad (3.8)$$

where $G_k(x)$ is the free outgoing propagator

$$G_k(x) = \frac{i}{2k} e^{ik|x|} \quad (3.9)$$

and equation (2.14) one obtains the first Born approximation

$$\begin{aligned} f_+^{(1)}(k) &= 2\epsilon R_{(1)}(\epsilon) - \frac{1}{2k} (2\epsilon W_{(1)}(\epsilon) + A_{(1)}(\epsilon) + B_{(1)}(\epsilon)) \\ f_-^{(1)}(k) &= -\frac{1}{2k} (2\epsilon W_{(1)}(\epsilon) + A_{(1)}(\epsilon) + B_{(1)}(\epsilon)) + i\epsilon (A_{(1)}(\epsilon) - B_{(1)}(\epsilon)) \\ &\quad + k\epsilon^2 \left(\frac{2}{3}\epsilon W_{(1)}(\epsilon) + A_{(1)}(\epsilon) + B_{(1)}(\epsilon) \right) + O(k^2). \end{aligned} \quad (3.10)$$

This implies, comparing with (3.7), that in the $\epsilon \rightarrow 0$ limit

$$\begin{aligned} R_{(1)}(\epsilon) &= -\frac{\theta}{2\epsilon} \\ 2\epsilon W_{(1)}(\epsilon) + A_{(1)}(\epsilon) + B_{(1)}(\epsilon) &= \alpha + \beta \\ \epsilon (A_{(1)}(\epsilon) - B_{(1)}(\epsilon)) &= 0 \end{aligned} \quad (3.11)$$

but the term linear in k in (3.7) cannot be obtained for $f_+(k)$ from (3.10). We then will have to consider it a second-order term and thus

$$\epsilon^2 \left(\frac{2}{3}\epsilon W_{(1)}(\epsilon) + A_{(1)}(\epsilon) + B_{(1)}(\epsilon) \right) = 0 \quad (3.12)$$

holds furthermore. Equations (3.11) and (3.12) have solutions for $W(\epsilon)$, $R(\epsilon)$, $A(\epsilon)$ and $B(\epsilon)$ to first order.

The second Born approximation gives, for the real part,

$$\begin{aligned} \operatorname{Re} f_+^{(2)}(k) &= 2\epsilon R_{(2)}(\epsilon) - \frac{1}{2k} (2\epsilon W_{(2)}(\epsilon) + A_{(2)}(\epsilon) + B_{(2)}(\epsilon)) \\ &\quad + \frac{\epsilon}{k} \left(R_{(1)}^2(\epsilon) - \frac{2}{3} \epsilon^2 W_{(1)}^2(\epsilon) - \epsilon W_{(1)}(\epsilon) (A_{(1)}(\epsilon) + B_{(1)}(\epsilon)) - A_{(1)}(\epsilon) B_{(1)}(\epsilon) \right) \\ &\quad + \frac{4}{3} k \epsilon^3 \left(\frac{2}{5} \epsilon^2 W_{(1)}^2(\epsilon) + \epsilon W_{(1)}(\epsilon) (A_{(1)}(\epsilon) + B_{(1)}(\epsilon)) + 2A_{(1)}(\epsilon) B_{(1)}(\epsilon) \right) \\ &\quad + O(k^2) \end{aligned} \quad (3.13)$$

and

$$\begin{aligned} \operatorname{Re} f_-^{(2)}(k) &= -\frac{1}{2k} (2\epsilon W_{(2)}(\epsilon) + A_{(2)}(\epsilon) + B_{(2)}(\epsilon)) - \frac{\epsilon}{2k} \left(\frac{4}{3} \epsilon^2 W_{(1)}^2(\epsilon) - 2R_{(1)}^2(\epsilon) \right. \\ &\quad \left. + 4\epsilon W_{(1)}(\epsilon) B_{(1)}(\epsilon) + B_{(1)}^2(\epsilon) - A_{(1)}^2(\epsilon) + 2A_{(1)}(\epsilon) B_{(1)}(\epsilon) \right) \\ &\quad + k \epsilon^2 \left(\frac{2}{3} \epsilon W_{(2)}(\epsilon) + A_{(2)}(\epsilon) + B_{(2)}(\epsilon) \right) \\ &\quad + \frac{1}{3} k \epsilon^3 \left(\frac{4}{5} \epsilon^2 W_{(1)}^2(\epsilon) - 2R_{(1)}^2(\epsilon) + 4\epsilon W_{(1)}(\epsilon) B_{(1)}(\epsilon) \right. \\ &\quad \left. + B_{(1)}^2(\epsilon) - A_{(1)}^2(\epsilon) + 2A_{(1)}(\epsilon) B_{(1)}(\epsilon) \right) + O(k^2) \end{aligned} \quad (3.14)$$

where the possible second-order counterterms of the first Born approximation have been included, as corresponds to a renormalizable perturbation theory. As the second order contributions to the scattering amplitudes are purely imaginary, (3.13) and (3.14) are bound to just reproduce the terms linear in k of (3.7). This implies, in the $\epsilon \rightarrow 0$ limit

$$\begin{aligned} \epsilon R_{(2)}(\epsilon) &= 0 \\ \frac{1}{2} (2\epsilon W_{(2)}(\epsilon) + A_{(2)}(\epsilon) + B_{(2)}(\epsilon)) \\ &= \epsilon \left(R_{(1)}^2(\epsilon) - \frac{2}{3} \epsilon^2 W_{(1)}^2(\epsilon) - \epsilon W_{(1)}(\epsilon) (A_{(1)}(\epsilon) + B_{(1)}(\epsilon)) \right. \\ &\quad \left. - A_{(1)}(\epsilon) B_{(1)}(\epsilon) \right) \end{aligned} \quad (3.15)$$

$$\frac{4}{3} \epsilon^3 \left(\frac{2}{5} \epsilon^2 W_{(1)}^2(\epsilon) + \epsilon W_{(1)}(\epsilon) (A_{(1)}(\epsilon) + B_{(1)}(\epsilon)) + 2A_{(1)}(\epsilon) B_{(1)}(\epsilon) \right) = \frac{1}{2\rho}$$

from (3.13) and

$$\begin{aligned}
& (2\epsilon W_{(2)}(\epsilon) + A_{(2)}(\epsilon) + B_{(2)}(\epsilon)) \\
&= -\epsilon \left(\frac{4}{3}\epsilon^2 W_{(1)}^2(\epsilon) - 2R_{(1)}^2(\epsilon) + 4\epsilon W_{(1)}(\epsilon)B_{(1)}(\epsilon) + B_{(1)}^2(\epsilon) \right. \\
&\quad \left. - A_{(1)}^2(\epsilon) + 2A_{(1)}(\epsilon)B_{(1)}(\epsilon) \right) \\
-\frac{1}{2\rho} &= \epsilon^2 \left(\frac{2}{3}\epsilon W_{(2)}(\epsilon) + A_{(2)}(\epsilon) + B_{(2)}(\epsilon) \right) \\
&\quad + \frac{1}{3}\epsilon^3 \left(\frac{4}{5}\epsilon^2 W_{(1)}^2(\epsilon) - 2R_{(1)}^2(\epsilon) + 4\epsilon W_{(1)}(\epsilon)B_{(1)}(\epsilon) \right. \\
&\quad \left. + B_{(1)}^2(\epsilon) - A_{(1)}^2(\epsilon) + 2A_{(1)}(\epsilon)B_{(1)}(\epsilon) \right)
\end{aligned} \tag{3.16}$$

from (3.14). One can convince oneself that the system given by equations (3.11), (3.12), (3.15) and (3.16) has no solution. Perturbation theory starting from equation (3.2) does not work; it is not renormalizable.

One can understand why. Perturbation theory in $1/\rho$ is based on the fact that the theory is free for large ρ (and $\alpha = \beta = \theta = 0$). Recall, however, that the exact regulated potential whose form we are using in this perturbative approach gives the correct large- ρ behaviour if first the regulator is removed taking $\epsilon \rightarrow 0$. But perturbation theory is taking these limits in reverse order, it assumes large ρ for fixed ϵ . The non-renormalizability just reflects the non-commutativity of the $\rho \rightarrow \infty$, $\epsilon \rightarrow 0$ limits.

A tell-tale sign that perturbation theory is not renormalizable is given by the fact that the theory is not asymptotically free, in other words, that it interacts even at infinite energy. This is saying that the potential diverges very strongly (stronger than a Dirac delta) as $\epsilon \rightarrow 0$. It is too strong to allow iterations as are done in perturbation theory.

This leads to the question of whether we are facing a genuinely non-perturbative problem, like tunnelling, or whether a perturbative approach starting from a different decomposition of unperturbed Hamiltonian and perturbation would allow a meaningful perturbation theory. Let us show that the second scenario holds.

4. The weak regulated potential

We have conjectured that asymptotic freedom is essential for perturbative renormalizability in quantum mechanics. This leads us to trying to shift the high-energy scattering into the free part of the Hamiltonian, so that the new interaction is asymptotically free. If our conjecture is correct this should then allow a regularization which works both exactly and perturbatively. This is indeed the case. Let us show how.

If the high-energy behaviour of the scattering amplitudes, equation (2.17), is subtracted, equation (2.14) reads

$$\psi(x) = 2 \cos(kx)\theta(-x) + ie^{ik|x|}(f_s(k) + (f_a(k) - i)\epsilon(x)) \tag{4.1}$$

where $\epsilon(x) \equiv \theta(x) - \theta(-x)$ and (2.25) has been used. Equation (4.1) is not a good starting point for perturbation theory, because the unperturbed wave vanishes on \mathbb{R}^+ . Let us therefore transform (4.1) under parity and under the substitution of (2.4):

$$\tilde{\psi}(-x) = 2 \cos(kx)\theta(x) + ie^{ik|x|}(\tilde{f}_s(k) - (\tilde{f}_a(k) - i)\epsilon(x)) \tag{4.2}$$

where $\tilde{f}(\rho, \alpha, \beta, \theta) = f(\rho, \beta, \alpha, -\theta)$. By summing and subtracting (4.1) and (4.2) one obtains

$$\psi^{(+)}(x) = \cos(kx) + ie^{ik|x|}(f_s^{(+)}(k) + f_a^{(+)}(k)\epsilon(x)) \quad (4.3)$$

$$\psi^{(-)}(x) = -\cos(kx)\epsilon(x) + ie^{ik|x|}(f_s^{(-)}(k) + f_a^{(-)}(k)\epsilon(x)) \quad (4.4)$$

where

$$\begin{aligned} f_s^{(+)}(k) &\equiv \frac{1}{2}(f_s(k) + \tilde{f}_s(k)) = \frac{1}{\Delta} \left(k\rho(\cos\theta - 1) - \frac{k}{2}(\alpha + \beta) - i\rho(\alpha + \beta) - i\alpha\beta \right) \\ f_a^{(+)}(k) &\equiv \frac{1}{2}(f_a(k) - \tilde{f}_a(k)) = \frac{k}{\Delta} \left(-i\rho \sin\theta + \frac{1}{2}(\alpha - \beta) \right) \\ f_s^{(-)}(k) &\equiv \frac{1}{2}(f_s(k) - \tilde{f}_s(k)) = \frac{k}{\Delta} \left(-i\rho \sin\theta - \frac{1}{2}(\alpha - \beta) \right) \\ f_a^{(-)}(k) &\equiv \frac{1}{2}(f_a(k) + \tilde{f}_a(k)) - i = \frac{1}{\Delta} \left(k\rho(\cos\theta + 1) + \frac{k}{2}(\alpha + \beta) + i\rho(\alpha + \beta) + i\alpha\beta \right). \end{aligned} \quad (4.5)$$

Note that the new scattering amplitudes all vanish at high energies. The new unperturbed solutions are characterized by having vanishing derivatives at the origin, as corresponds to $\rho = \alpha = \beta = \theta = 0$; recall equation (2.6).

Looking at the wavefunctions as given by equations (4.3) and (4.4) for each parameter alone we note that they correspond to δ -interactions. So we expect that Dirac deltas appear in some sense in the potential.

We will therefore regularize the interaction with Dirac deltas away from the origin. Having in mind that Dirac deltas can be characterized in terms of a boundary condition, as shown in (2.10), which we write as

$$\Delta\psi'(0) = \lambda\psi(0) \quad \longleftrightarrow \quad \lambda\delta(x)\psi(x) \quad (4.6)$$

we now rewrite equation (2.1) in a form which resembles (4.6). In order to do so, let us split it into conditions on two boundaries, writing $\psi_L \rightarrow \psi(-\epsilon)$, $\psi_R \rightarrow \psi(\epsilon)$, $\psi'_L \rightarrow -\Delta\psi'(-\epsilon)$ and $\psi'_R \rightarrow \Delta\psi'(\epsilon)$, where we have taken $\psi'(0) = 0$. Then equation (2.1) is substituted by

$$\begin{aligned} \Delta\psi'(-\epsilon) &= (\rho + \alpha)\psi(-\epsilon) - \rho e^{i\theta}\psi(\epsilon) \\ \Delta\psi'(\epsilon) &= -\rho e^{-i\theta}\psi(-\epsilon) + (\rho + \beta)\psi(\epsilon). \end{aligned} \quad (4.7)$$

In the limit $\epsilon \rightarrow 0$ equation (4.7) coincides with (2.1)

Comparing equations (4.7) and (4.6) immediately leads to the Schrödinger equation

$$\begin{aligned} -\frac{d^2}{dx^2}\psi(x) + ((\rho + \beta)\delta(x - \epsilon) + (\rho + \alpha)\delta(x + \epsilon))\psi(x) \\ + (-\rho e^{-i\theta}\delta(x - \epsilon) - \rho e^{i\theta}\delta(x + \epsilon))\psi(-x) = E\psi(x) \end{aligned} \quad (4.8)$$

subject to the boundary condition

$$\psi'(0) = 0. \quad (4.9)$$

Note that (4.8) is non-local. This non-locality is, however, avoided by just rewriting (4.8) in terms of the parity eigenfunctions, i.e. $\psi_s(x) \equiv \frac{1}{2}(\psi(x) + \psi(-x))$ and $\psi_a(x) \equiv \frac{1}{2}(\psi(x) - \psi(-x))$. Then equation (4.8) becomes

$$\left[-\frac{d^2}{dx^2} + \begin{pmatrix} V_{ss}(x) & V_{sa}(x) \\ V_{as}(x) & V_{aa}(x) \end{pmatrix} \right] \begin{pmatrix} \psi_s(x) \\ \psi_a(x) \end{pmatrix} = E \begin{pmatrix} \psi_s(x) \\ \psi_a(x) \end{pmatrix} \quad (4.10)$$

where the potentials are given by

$$\begin{aligned} V_{ss}(x) &= \frac{1}{2}(\alpha + \beta + 2\rho(1 - \cos\theta))(\delta(x - \epsilon) + \delta(x + \epsilon)) \\ V_{aa}(x) &= \frac{1}{2}(\alpha + \beta + 2\rho(1 + \cos\theta))(\delta(x - \epsilon) + \delta(x + \epsilon)) \\ V_{sa}(x) &= -\frac{1}{2}(\alpha - \beta + 2i\rho \sin\theta)(\delta(x - \epsilon) - \delta(x + \epsilon)) = V_{as}^*(x) \end{aligned} \quad (4.11)$$

together with

$$\psi'_s(0) = \psi'_a(0) = 0. \quad (4.12)$$

The potential matrix has been written in such a way that $V_{ss}(x)$ and $V_{aa}(x)$ are symmetric and $V_{sa}(x) = V_{as}^*(x)$ are antisymmetric. The solutions of equation (4.10) reproduces (4.3) and (4.4) in the limit $\epsilon \rightarrow 0$.

The interaction regulated in this way is actually a problem in the Hilbert space $L^2(\mathbb{R}^+ \oplus \mathbb{R}^-)$ instead of $L^2(\mathbb{R})$ with potentials acting on each halfline. The complex potentials are responsible for the probability flow between the halflines. This topological feature of the contact interaction is of course unique to one dimension, it does not happen in higher dimensions.

One can rewrite equation (4.8) with the help of the parity operator, \mathbb{P} , and equation (4.11) as

$$\begin{aligned} -\frac{d^2}{dx^2}\psi(x) + \frac{1}{2}[(V_{ss}(x) + V_{aa}(x)) + (V_{as}(x) + V_{sa}(x))]\psi(x) \\ + \frac{1}{2}[(V_{ss}(x) - V_{aa}(x)) + (V_{as}(x) - V_{sa}(x))]\mathbb{P}\psi(x) = E\psi(x). \end{aligned} \quad (4.13)$$

It might seem surprising that the parity operator now appears in the regulated potential, whereas the momentum operator appeared in (3.1). This is because the momentum operator is not regularized when acting together with Dirac deltas and if the Dirac deltas are further regularized, then it would act at the origin, where the wavefunction is allowed to be discontinuous. The parity operator avoids these problems and still allows violation of time reversal invariance, as can be seen from the term $(V_{as}(x) - V_{sa}(x))\mathbb{P}$, which is imaginary.

To perform perturbation theory starting from the Lippmann–Schwinger equation we need the propagator corresponding to the unperturbed solutions of (4.3) and (4.4)

$$G_k(x, y) = \frac{i}{2k} (e^{ik|x+y|} + e^{ik|x-y|}) (\theta(x)\theta(y) + \theta(-x)\theta(-y)) \quad (4.14)$$

which represents outgoing propagation that does not cross the origin and that satisfies the adequate boundary conditions at the origin, i.e. vanishing derivatives.

Using equations (4.11) and (4.14) one writes the Lippmann–Schwinger equation for $\psi^{(+)}(x)$

$$\begin{pmatrix} \psi_s^{(+)}(x) \\ \psi_a^{(+)}(x) \end{pmatrix} = \begin{pmatrix} \cos(kx) \\ 0 \end{pmatrix} - \int dy G_k(x, y) \begin{pmatrix} V_{ss}(y) & V_{sa}(y) \\ V_{as}(y) & V_{aa}(y) \end{pmatrix} \begin{pmatrix} \psi_s^{(+)}(y) \\ \psi_a^{(+)}(y) \end{pmatrix} \quad (4.15)$$

which allows us to perform the perturbative expansion in a straightforward way. This expansion can immediately be summed, and it reproduces the exact result (4.3) in the limit $\epsilon \rightarrow 0$. The same can be done for $\psi^{(-)}(x)$ with the same potential and we obtain the result of (4.4).

Note that for the $\delta', \alpha = \beta = \theta = 0$, only $V_{aa}(x)$ remains. It is symmetric but only acts on the odd wavefunctions.

The four-parameter contact interaction is finally very elementary, as seen in equation (4.11), once the unperturbed Hamiltonian and its accompanying boundary conditions are properly chosen.

5. Conclusion

The most general point interaction in one-dimensional quantum mechanics depends on four real parameters which determine the boundary conditions at the site of the interaction. We present here, for the first time, a regulated potential which leads to a self-adjoint Hamiltonian and which reproduces the same physics when the regulator is removed. Perturbation theory built upon this regulated potential is, however, not renormalizable. The regulator cannot be removed. We conjecture that this is because the interaction is not asymptotically free, i.e. because there exists scattering at infinite energy. This reflects too strong a potential, which thus cannot be iterated perturbatively.

We then present a different breakup of unperturbed Hamiltonian and interaction, for which the interaction is asymptotically free. We give its regulated form, which allows both the exact and the perturbative solution. This provides support to our conjecture.

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